Non-radiative synthesis of ^{7}Be in solar plasma

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Abstract

The nuclear reaction $e+{}^3\mathrm{He}+{}^4\mathrm{He}\to e+{}^7\mathrm{Be}$ is considered at thermonuclear energies. The motion of the electron is treated within the adiabatic approximation and the ${}^3\mathrm{He}-{}^4\mathrm{He}$ scattering state is calculated using an effective ${}^3\mathrm{He}-{}^4\mathrm{He}$ potential constructed via the Marchenko inverse scattering method. The reaction rate thus obtained for solar interior conditions is approximately 10^{-4} of the corresponding rate for the radiative capture ${}^3\mathrm{He}+{}^4\mathrm{He}\to{}^7\mathrm{Be}+\gamma$. However, at the conditions corresponding to the early stages of the primordial nucleosynthesis, the non-radiative fusion is five times faster than the radiative one. The importance of the non-radiative processes for the energy balance in the sun, which for electrons is totally opaque, is discussed.

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I. INTRODUCTION

In the standard model of the sun it is assumed that all nuclear reactions forming the pp–chain, are generated by various two–body nucleus–nucleus and electron–nucleus collisions with only one exception, namely, the $e+p+p\to d+\nu$ reaction [1]. Other three–body reactions are ignored on the grounds that the triple collisions occur less frequently than the binary collisions.

Defining the collision as a situation in which the particles approach each other to a distance less than the range R of their interaction, then the frequency of two–body collisions per unit volume is

$$\mathcal{F}_{12}=\pi R^2 v_{12} n_1 n_2\ ,$$

where v_{12} is the relative velocity of particles 1 and 2, and n_i are their densities. The frequency of the triple collision $\mathcal{F}_{123}$ is obtained if we multiply $\mathcal{F}_{12}$ by the probability of finding the particle 3 present when 1 and 2 are scatterred [2], viz. by $(4\pi/3)R^3 n_3$. Then

$$\mathcal{F}_{123} = \frac{4\pi^2}{3} R^5 v_{12} n_1 n_2 n_3 \ .$$

From these elementary considerations one can find that triple collisions are less probable than the binary ones. For example, in the centre of the sun the densities of electrons, ^{3}He, and ^{4}He nuclei are (these can be derived from the Tables 4.4 and 4.5 of Ref. [1])

$$\begin{aligned} n_e &= 0.60 \times 10^{26} \ \mathrm{cm}^{-3} \ , \\ n_{^3\mathrm{He}} &= 0.23 \times 10^{21} \ \mathrm{cm}^{-3} \ , \\ n_{^4\mathrm{He}} &= 0.14 \times 10^{26} \ \mathrm{cm}^{-3} \ , \end{aligned} \tag{1}$$

which together with a nuclear reaction range of $R \sim 10\,\mathrm{fm}$ and a velocity of $v_{^3\mathrm{He}^4\mathrm{He}} \sim 3 \times 10^7\,\mathrm{cm/sec}$ corresponding to the temperature $1.56 \times 10^7\ {}^\circ\mathrm{K}$, give

$$\begin{aligned} \mathcal{F}_{^3\mathrm{He}^4\mathrm{He}} &\sim 0.3 \times 10^{30} \ \mathrm{cm}^{-3}\mathrm{sec}^{-1} \ , \\ \mathcal{F}_{e^3\mathrm{He}^4\mathrm{He}} &\sim 0.8 \times 10^{20} \ \mathrm{cm}^{-3}\mathrm{sec}^{-1} \ . \end{aligned}$$

This, however, is not a sufficient proof that the three–body reactions can be ignored. This is due to the fact that not every collision results in a nuclear transition. The actual reaction rate depends also on the transition probability which may be higher for three–body rather than for two–body initial states. This stems from the fact that triple collisions are kinematically less restricted than the binary ones. Some binary reactions are forbidden by conservation laws, e.g., conservation of angular momentum, parity, isospin etc. However, such reactions could take place in the presence of a third particle.

Calculations of the three–body reaction rates performed so far [3–5], corroborate this by showing that the simplified considerations based on the frequency of collisions overestimate the differences between the two- and three-body reaction rates. Thus, for example, in Ref. [3] it was shown that under the conditions which prevailed during the primordial nucleosynthesis, the non-radiative capture $e + n + p \to e + d$ occurs at a reaction rate which is five orders of magnitude less than the corresponding rate for the radiative capture $n+p \to \gamma+d$. In Refs. [4,5] the reaction rates for the non-radiative processes $e + p + d \to e + {}^3\mathrm{He}$ and $e + p + {}^7\mathrm{Be} \to e + {}^8\mathrm{B}$ were calculated. Again, it was found that they are only four orders of magnitude less than for corresponding binary reactions $p+d \to \gamma+{}^3\mathrm{He}$ and $p+{}^7\mathrm{Be} \to \gamma+{}^8\mathrm{B}$. In contrast, a simplified estimation provides a rate which differs by ten orders of magnitude from them.

The above examples clearly demonstrate that a reliable answer to the question of whether or not a particular three–body reaction is negligible, can be obtained only when proper calculations are made. And surely in the case under consideration, such calculations should be done as we have the ^{7}Be paradox on top of the long–standing solar neutrino problem. This paradox originated from the combined analysis of all experiments, in which the neutrino flux from the sun was measured. This analysis came to the conclusion that the production of ^{7}Be nuclei (more precisely, the flux of neutrinos due to ^{7}Be reactions) must be strongly suppressed or even be negative [6,7]. This implies that something is wrong either in the standard model or in the experimental data. In this respect the fate of ^{7}Be in the *pp*–chain is of special interest.

According to the standard model this nucleus is produced via the radiative fusion

$$ {}^3\mathrm{He} + {}^4\mathrm{He} \longrightarrow {}^7\mathrm{Be} + \gamma \ , \tag{2} $$

in which the released energy (1.59 MeV) is carried away by the photon. Another possibility for a ^{7}Be production is the three–body non-radiative fusion

$$ e + {}^3\mathrm{He} + {}^4\mathrm{He} \longrightarrow e + {}^7\mathrm{Be} \ . \tag{3} $$

This is a kind of Auger transition in which the same amount of the released energy is carried away by the electron.

As we have seen, the triple collisions (3) are less frequent than the binary ones (2) and thus one can argue that they are unimportant. We have, however, at least three reasons to perform explicit calculations for the reaction rate of the process (3). Firstly, since the neutrino paradox persists we have to be scrupulous in examining any conjecture on the matter. Secondly, although the three-body reaction may be unimportant in the neutrino problem, it may contribute to the energy balance in the sun since the energy released in each non-radiative fusion event remains in the center of the sun because the electron which carries it, cannot escape. In contrast to photons, for electrons the sun is totally opaque. To estimate this additional heating, we need to know the three–body reaction rate. Finally, at the early stages of the primordial nucleosynthesis the temperature as well as the particle densities were much higher than they are in the solar plasma. We therefore may expect that the three–body reactions were important at that times.

The paper is organized as follows. In Sec. II we describe our formalism in general and in Sec. III we outline the procedure and approximations employed to evaluate the transition operator. Sec. IV is devoted to the potentials we use to describe the interactions among the particles, and In Sec. V we present our results and conclusions.

II. THREE–BODY REACTION RATE

The nuclei ^{3}He and ^{4}He are stable clusters (in what follows we denote them by τ and α respectively) in the low–lying states of ^{7}Be (see Ref. [8] and references therein). We can, therefore, reduce the eight–body problem $e(NNN)(NNNN)$ to the three–body one, namely, $e-\tau-\alpha$. The corresponding Jacobi vectors in configuration and momentum spaces are shown in Fig. 1.

Let $\mathcal{R}(\mathbf{k},\mathbf{p}\to\mathbf{p}')$ be the reaction rate per unit volume per second for the collision process (3) which starts with the momenta $(\mathbf{k},\mathbf{p})$ and ends up with $\mathbf{p}'$. The plasma particles are in a thermodynamical equilibrium and their momenta are distributed according to Maxwell's law

$$ N_{\mathbf{k}}(\theta) = (2\pi\mu\kappa\theta)^{-3/2} \exp\left(-\frac{k^2}{2\mu\kappa\theta}\right) \ , $$
$$ N_{\mathbf{p}}(\theta) = (2\pi m\kappa\theta)^{-3/2} \exp\left(-\frac{p^2}{2m\kappa\theta}\right) \ , $$

where $N_{\mathbf{k}}$ and $N_{\mathbf{p}}$ are the probability densities, μ is the $\tau\alpha$–reduced mass, m is the electron mass, κ is the Boltzmann constant, and θ is the plasma temperature. We are cocerned with the total rate of the transition from an initial state with any $(\mathbf{k},\mathbf{p})$ to a final state with all

possible $\mathbf{p}'$. Thus the reaction rate $\mathcal{R}(\mathbf{k},\mathbf{p}\to\mathbf{p}')$ must be averaged over the initial momenta $\mathbf{k}$ and $\mathbf{p}$ and integrated over the final momentum $\mathbf{p}'$, i.e.,

$$\langle\mathcal{R}\rangle_\theta = \int\int\int d\mathbf{k}d\mathbf{p}d\mathbf{p}'\,\mathcal{R}(\mathbf{k},\mathbf{p}\to\mathbf{p}')N_{\mathbf{k}}(\theta)N_{\mathbf{p}}(\theta)\,. \tag{4}$$

Similarly to the two–body reaction theory where the average reaction rate $\langle\mathcal{R}_{12}\rangle_\theta$ is written as a product of $\langle\sigma_{12}v_{12}\rangle_\theta$ (which is referred to as the reaction rate per particle pair) and the particle densities [9],

$$\langle\mathcal{R}_{12}\rangle_\theta = n_1n_2\langle\sigma_{12}v_{12}\rangle_\theta\ ,$$

the three–body reaction rate $\langle\mathcal{R}\rangle_\theta$ can also be factorized in the same manner,

$$\langle\mathcal{R}\rangle_\theta = n_e n_\tau n_\alpha\langle\Sigma\rangle_\theta\ . \tag{5}$$

Indeed, using the general formula for a three–body reaction rate [2] we obtain

$$\begin{aligned}\mathcal{R}\left(\mathbf{k},\mathbf{p}\to\mathbf{p}'\right) = \frac{(2\pi)^7}{2}n_e n_\tau n_\alpha \sum_{nm_sm_j}\delta\left(\frac{p'^{\,2}}{2m}-E^{(n)}-\frac{k^2}{2\mu}-\frac{p^2}{2m}\right)\\ \times\,|\,\langle\psi^{(n)}_{7\,jm_j};\mathbf{p}'\,|\,T\,|\,\mathbf{k},m_s;\mathbf{p}\rangle|^2\ ,\end{aligned} \tag{6}$$

where the δ–function secures the energy conservation for the transition to the n-th level of ^{7}Be nucleus (see Fig. 2) accompanied by the energy release $E^{(n)}$ and T is the transition operator for the process (3). The initial and final asymptotic state wave functions for this reaction, $|\mathbf{k},m_s;\mathbf{p}\rangle$ and $|\psi^{(n)}_{7\,jm_j};\mathbf{p}'\rangle$, are characterized, apart from the momenta, by the nuclear spin $s=1/2$ of the ^{3}He, the total angular momentum j of ^{7}Be (which depends on the state n), and their third components m_s and m_j respectively. They are normalized as

$$\langle\,\mathbf{k}',m_s';\mathbf{p}'\mid\mathbf{k},m_s;\mathbf{p}\rangle = \delta_{m_s'm_s}\delta(\mathbf{k}-\mathbf{k}')\ \delta(\mathbf{p}-\mathbf{p}')\ ,$$

$$\langle\,\psi^{(n')}_{7\,jm_j'};\mathbf{p}'\mid\psi^{(n)}_{7\,jm_j};\mathbf{p}\rangle = \delta_{n'n}\,\delta_{m_j'm_j}\,\delta(\mathbf{p}-\mathbf{p}')\ .$$

The sum $\frac{1}{2}\sum_{m_sm_j}$ in (6) is for averaging over the initial and summing over the final spin orientations. Therefore the quantity $\langle\Sigma\rangle_\theta$ defined by Eq. (5), in analogy to the two–body case, may be referred to as the reaction rate per particle trio and is given by

$$\begin{aligned}\langle\Sigma\rangle_\theta = \frac{(2\pi)^7}{2}\sum_{nm_sm_j}\int\int\int d\mathbf{k}d\mathbf{p}d\mathbf{p}'\,\delta\left(\frac{p'^{\,2}}{2m}-E^{(n)}-\frac{k^2}{2\mu}-\frac{p^2}{2m}\right)\\ \times|\,\langle\psi^{(n)}_{7\,jm_j};\mathbf{p}'\,|\,T\,|\,\mathbf{k},m_s;\mathbf{p}\rangle\,|^2\,N_{\mathbf{k}}(\theta)N_{\mathbf{p}}(\theta)\,.\end{aligned} \tag{7}$$

In the next two sections we describe how the various ingredients needed to obtain the matrix element $\langle\psi^{(n)}_{7\,jm_j};\mathbf{p}'\,|\,T\,|\,\mathbf{k},m_s;\mathbf{p}\rangle$ were calculated.

III. TRANSITION OPERATOR

The three–body quantum state $|e + {}^3\mathrm{He} + {}^4\mathrm{He}\rangle$ belongs to the Hamiltonian

$$H = H_0 + h_0 + V_N + V_e\,, \tag{8}$$

where H_0 and h_0 are the kinetic energy operators associated with the Jacobi variables $\mathbf{r}$ and $\boldsymbol{\rho}$ respectively (see Fig. 1),

$$V_N = V^{(s)}_{\tau\alpha} + V^{(c)}_{\tau\alpha}$$

is the nuclear τ-α potential, which includes strong and Coulombic parts. The V_e consists of the e-τ and e-α Coulomb potentials

$$V_e = V_{e\tau} + V_{e\alpha}\ .$$

The various transitions in this three–body system are determined by the relevant matrix elements of the T–operator obeying the Lippmann–Schwinger equation

$$T(z) = V_N + V_e + (V_N + V_e)\frac{1}{z - H_0 - h_0}T(z)\,. \tag{9}$$

The probability of the transition (3), i.e. of

$$|\mathbf{k}, m_s; \mathbf{p}\rangle \ \stackrel{V_N+V_e}{\Longrightarrow}\ \left|\psi^{(n)}_{7\,jm_j}; \mathbf{p}'\right\rangle\ , \tag{10}$$

is therefore determined from the matrix element $\langle\psi^{(n)}_{7\,jm_j}; \mathbf{p}'\,|\,T\,|\,\mathbf{k}, m_s; \mathbf{p}\rangle$.

Making use of the special conditions prevailing in solar plasma and the smallness of the electron mass, we can significantly simplify Eq. (9). Indeed, the average kinetic energy of the particles in the plasma, $< E^{kin} >\sim \kappa\theta \approx 1\,\mathrm{keV}$, is the same for nuclei and electrons, but the velocity of an electron is three orders of magnitude higher than that of ^{3}He or ^{4}He. Therefore, while the τ approaches α very slowly, the electron dashes nearby picking up the energy and leaving the heavy particles in a bound state. Due to this we can treat the relative $\tau\alpha$ motion adiabatically. For a three–body collision such an approximation results in two simplifications. In the first one the action of the potentials V_N and V_e can be separated. Indeed, while the electron starts from its asymptotic state $|\mathbf{p}\rangle$, the heavy particles have already interacted via the potential V_N and formed the two–body scattering state

$$|\mathbf{k}, m_s\rangle \ \stackrel{V_N}{\Longrightarrow}\ |\psi_{\mathbf{k},m_s}\rangle\ . \tag{11}$$

Therefore, instead of the transition (10) caused by both V_N and V_e, in the adiabatic approximation we may consider the transition

$$|\psi_{\mathbf{k},m_s}; \mathbf{p}\rangle \ \stackrel{V_e}{\Longrightarrow}\ \left|\psi^{(n)}_{7\,jm_j}; \mathbf{p}'\right\rangle\ , \tag{12}$$

where the interaction V_N is taken into account by (11). In other words, the transition (10) effectively occurs in two steps, (11) and (12). As a result the two–body scattering problem (11) can be solved separately.

In the second simplification the transition (12) may be described using the fixed scatterer T–matrix defined as

$$\tilde{T}(z) = V_e + V_e \frac{1}{z - h_0} \tilde{T}(z) \ . \tag{13}$$

Within the above approximation we have

$$\langle \psi^{(n)}_{7\,jm_j}; \mathbf{p}' | T | \mathbf{k}, m_s; \mathbf{p} \rangle \approx \langle \psi^{(n)}_{7\,jm_j}; \mathbf{p}' | \tilde{T} | \psi_{\mathbf{k},m_s}; \mathbf{p} \rangle \ . \tag{14}$$

A further simplification can be achieved when Eq. (13) is solved iteratively. For the solar plasma electrons one has the condition $e^2/hv < 1$, which is a sufficient to treat Coulomb interactions in Eq. (13) perturbatively, i.e.,

$$\tilde{T}(z) = V_e + V_e \frac{1}{z - h_0} V_e + V_e \frac{1}{z - h_0} V_e \frac{1}{z - h_0} V_e + \cdots \ ,$$

Furthermore, the average potential energy of the electron is of atomic order of magnitude, $< V_e > \sim 10$ eV, while its kinetic energy in solar plasma is two orders of magnitude higher, $< E_e^{kin} > \sim 1$ keV, which implies that the above iterations should converge very fast. Therefore, we may retain only the first (Born) term [10].

Finally, the exact matrix elements of the T–operator can be replaced by the approximate ones

$$\langle \psi^{(n)}_{7\,jm_j}; \mathbf{p}' \,|\, T \,|\, \mathbf{k}, m_s; \mathbf{p} \rangle \approx \langle \psi^{(n)}_{7\,jm_j}; \mathbf{p}' \,|\, V_e \,|\, \psi_{\mathbf{k},m_s}; \mathbf{p} \rangle \ . \tag{15}$$

This can be considered as a three-body generalization of the distorted wave Born approximation (DWBA) which is widely used in the theory of the two–body nuclear reactions.

IV. POTENTIALS

The evaluation of the matrix element (15) requires the knowledge of the potential V_e and the wave functions $\psi_{\mathbf{k},m_s}(\mathbf{r})$ and $\psi^{(n)}_{7\,jm_j}(\mathbf{r})$. In order to obtain the wave functions we solved the $\tau\alpha$ two–body problem using the Jost function method, proposed in Refs. [15,16], with the $\tau\alpha$ effective potential $V_{\tau\alpha} = V^{(s)}_{\tau\alpha} + V^{(c)}_{\tau\alpha}$.

A. Coulomb forces

The electron as a point-like particle generates a pure Coulomb field. However, the electric charges of τ and α are distributed within their nuclear volumes which have a size comparable to the typical range of the nuclear reaction (few fm). Furthermore, the average collision energy, $\sim 1\,$keV, is comparable to the height of the Coulomb barriers and thus the charge distributions should be taken into account. This is achieved by assuming that the nuclear charge distributions are spherically symmetric and have a Gaussian form

$$q_i(\varrho) = 2e \left(\frac{3}{2\pi \langle R_i^2 \rangle} \right)^{\frac{3}{2}} \exp\left(-\frac{3\varrho^2}{2\langle R_i^2 \rangle} \right) \ , \qquad i = \tau, \alpha \ , \tag{16}$$

where ϱ is the distance from the centre of the nucleus. The normalization coefficient and the slope parameter in (16) are chosen to give the total charges and the r.m.s. radii of the nuclei, i. e.

$$\int q_i(\varrho)\, d^3\varrho = 2e \ ,$$

$$\frac{1}{2e}\int \varrho^2 q_i(\varrho)\, d^3\varrho = \langle R_i^2\rangle \ .$$

For the r.m.s. radii we used the experimental values [11,12]

$$\sqrt{\langle R_\tau^2\rangle} = 1.93\,\mathrm{fm} \ , \qquad \sqrt{\langle R_\alpha^2\rangle} = 1.67\,\mathrm{fm} \ .$$

Using the space charge distribution (16), we obtain

$$V_{ei}(\varrho) = \frac{2e^2}{\varrho}\,\mathrm{erf}\left(\varrho\sqrt{\frac{3}{2\langle R_i^2\rangle}}\right) \ , \qquad i = \tau, \alpha \ , \tag{17}$$

$$V_{\tau\alpha}^{(c)}(\varrho) = \frac{4e^2}{\varrho}\sqrt{\frac{6}{\pi\langle R_\alpha^2\rangle}}\int_0^\infty \exp\left[-\frac{3(\varrho'^2+\varrho^2)}{2\langle R_\alpha^2\rangle}\right]\sinh\left(\frac{3}{\langle R_\alpha^2\rangle}\varrho\varrho'\right)\,\mathrm{erf}\left(\varrho'\sqrt{\frac{3}{2\langle R_\tau^2\rangle}}\right)\,d\varrho' \ . \tag{18}$$

At large distances this electron–nucleus potential (17) has the usual Coulomb tail $2e^2/\varrho$. At short distances, however, it is essentially different without having a singularity at $\varrho = 0$. The same can be said about the nucleus–nucleus interaction (18).

The electron which participates in the three–body reaction, is only one of the great number of the plasma electrons. All the others are spectators surrounding the nuclei and thus their combined Coulomb field reduces the electric fields of the nuclei. In the standard Debye–Hückel theory [13] such screening effect is taken into account by introducing an exponentially decaying factor into the Coulomb part of the nucleus–nucleus potential. Following this approach, we replace the potentials (17) and (18) by the screened ones

$$V_{ei}(\varrho) \ \stackrel{screen}{\longrightarrow}\ V_{ei}(\varrho)\exp\left(-\frac{\varrho}{D}\right) \ , \qquad i = \tau, \alpha \ , \tag{19}$$

$$V_{\tau\alpha}^{(c)}(\varrho) \ \stackrel{screen}{\longrightarrow}\ V_{\tau\alpha}^{(c)}(\varrho)\exp\left(-\frac{\varrho}{D}\right) \ , \tag{20}$$

with the Debye radius being $D = 21800\,\mathrm{fm}$ which corresponds to the solar plasma conditions and is typical for other stars as well [14].

B. Nuclear forces

As can be seen in Fig. 2, when ^{3}He and ^{4}He nuclei are fused in solar plasma, they may form either the ground or the first excited state of ^{7}Be, the quantum numbers j^π being $3/2^-$ and $1/2^-$ respectively. The rest of the excited states are situated very high and hence we can safely ignore the virtual transitions via them. Therefore, in constructing the nucleus–nucleus scattering wave function $\psi_{\mathbf{k},m_s}$ we assume that ^{3}He and ^{4}He may interact if their total angular momentum $j \le 3/2$. This restricts the allowable values of their relative orbital

momentum ℓ to $\ell \leq 2$. The state with $\ell = 2$ can also be ignored since at collision energies $\sim 1\,\mathrm{keV}$ the contribution from the higher partial waves diminishes very fast. Thus for the nuclear forces between ^{3}He and ^{4}He we consider those corresponding to the lowest three (ℓ, j)–states, namely, (0,1/2), (1,1/2), and (1,3/2).

To construct the corresponding $\tau\alpha$–potentials, we employ the Marchenko inverse scattering method [17,18] which is briefly outlined below. Within this method we obtained energy independent local potentials $V_{\tau\alpha}^{(s)}(r)$ for each of the above three partial waves, which reproduce the the Resonating Group Model (RGM) $\tau\alpha$–scattering phase–shifts [19,20] and give the correct binding energies for the ground and first excited states of ^{7}Be (in the $\tau\alpha$–model).

In the Marchenko inverse scattering method a unique, energy–independent, ℓ–dependent, local potential $V_\ell(r)$ is obtained from

$$V_\ell(r) = -2\frac{\mathrm{d}}{\mathrm{d}r}K_\ell(r,r) \tag{21}$$

where the kernel $K_\ell(r,r')$ obeys the Marchenko fundamental equation

$$K_\ell(r,r') + F_\ell(r,r') + \int_r^\infty K_\ell(r,r'')F_\ell(r'',r')dr'' = 0. \tag{22}$$

The driving term $F_\ell(r,r')$ is given by

$$F_\ell(r,r') = \frac{1}{2\pi}\int_{-\infty}^{+\infty} w_\ell^+(kr)[1-S_\ell(k)]w_\ell^+(kr')\mathrm{d}k + A_\ell w_\ell^+(b_\ell r)w_\ell^+(b_\ell r'). \tag{23}$$

$S_\ell(k)$ is the S–matrix for the specific partial wave, and the function $w_\ell^+(z)$ is related to the spherical Hankel function $h_\ell^{(+)}(z)$ by

$$w_\ell^+(z) = ie^{i\pi\ell}zh_\ell^{(+)}(z). \tag{24}$$

Furthermore, A_ℓ is the so–called asymptotic bound state normalisation constant, while $b_\ell = \sqrt{-2\mu E_b^\ell}$, E_b^ℓ being the $\tau\alpha$ bound state energy and μ their reduced mass.

The evaluation of $F_\ell(r,r')$ requires the knowledge of the S–matrix for all real energies from the elastic scattering threshold to infinity, together with the reflection property $S_\ell(-k) = 1/S_\ell(k)$, as well as the binding energies and the corresponding asymptotic bound state normalisation constants. It is greatly simplified by choosing a rational (Bargmann–type) parametrisation

$$S_\ell(k) = \prod_{m=1}^{N_B}\frac{k+ib_\ell^m}{k-ib_\ell^m}\prod_{n=1}^{N_\ell}\frac{k+\alpha_n^\ell}{k-\alpha_n^\ell} \tag{25}$$

where N_B is the number of bound states present. The number $N_\ell + N_B$ must be even to satisfy the properties of the S–matrix. The α_n^ℓ are complex numbers used to fit the (numerically) given S–matrix. We mention here that the so–called Pauli Forbidden States (PFS) are ignored. This implies that the resulting potentials are the unique supersymmetric shallow [21] partners sustaining only the physical bound states.

With the choice (25) the integration in Eq. (23) can be easily performed analytically (for more details see [18,21,22]).

The potentials $V_{\tau\alpha}^{(s)}$ thus obtained for the $\tau\alpha$ system in the channels $\mathrm{S}_{1/2}$, $\mathrm{P}_{1/2}$, and $\mathrm{P}_{3/2}$ are shown in Fig. 3.

V. RESULTS AND CONCLUSIONS

The plasma temperature in the center of the sun is $\theta_0 = 15.6 \times 10^6\,^\circ$K [1] and decreases, by a factor of ten towards the surface. Calculating the reaction rate for the process (3) we need, therefore, to cover this range of θ. The same *pp*–chain reactions occur in other stars as well with similar conditions. In the blue stars, which are hotter than the sun, the plasma temperature is much higher. Thus, it is worthwhile to extend our calculations beyond the solar temperatures too. Another reason for such an extension comes from the necessity to examine the role of the triple collisions in the primordial nucleosynthesis which also proceeded via the *pp*–cycle. The temperature at which the creation of the nuclei occured in the early universe was $\sim 10^9\,^\circ$K [9].

We, therefore, calculated the three–body reaction rate for the values of θ ranging from $1 \times 10^6\,^\circ$K to $3 \times 10^9\,^\circ$K. The results of our calculations are given in Table I. Since the reaction rate obtained can be used not only in models of the sun but also for other stellar objects as well where the plasma densities may be quite different, we present it in units of cm^{6}mole^{-2}sec^{-1} which are convenient for a general use [23]. The meaning of these units is that instead of $n_e n_\tau n_\alpha$ we multiply $\langle\Sigma\rangle$ by N_A^2, i.e. the Avogadro number squared. Then for any specific densities the reaction rate, in the units cm^{-3}sec^{-1}, can be obtained from Table I by multiplying the values given there, by $n_e n_\tau n_\alpha / N_A^2$.

To answer the question of the role played by the three–body reaction (3) in the solar *pp*–cycle, we need to compare its reaction rate with the corresponding rate of the two–body process (2). Using the particle densities (1), we obtain for the center of the sun

$$\langle\mathcal{R}_{e\tau\alpha}\rangle_{\theta_0} = 7.15 \times 10^3 \text{ cm}^{-3}\text{sec}^{-1} \ ,$$

while for the corresponding two–body process [23]

$$\langle\mathcal{R}_{\tau\alpha}\rangle_{\theta_0} = 2.44 \times 10^7 \text{ cm}^{-3}\text{sec}^{-1} \ .$$

Their ratio is rather small,

$$\frac{\langle\mathcal{R}_{e\tau\alpha}\rangle_{\theta_0}}{\langle\mathcal{R}_{\tau\alpha}\rangle_{\theta_0}} = 2.94 \times 10^{-4} \ , \tag{26}$$

but not as small as one would guess from the simplified considerations discussed in the introduction. The reaction rate $\langle\mathcal{R}_{e\tau\alpha}\rangle_\theta$ in other parts of the sun together with its ratio to the two–body reaction rate are given in Table II. In obtaining these data, we used the dependences of n_e, n_τ, n_α, and θ on the distace R from the solar centre given in Ref. [1].

When we move away from the center of the sun the temperature and the electron density decrease. As a result the ratio $\langle\mathcal{R}_{e\tau\alpha}\rangle_\theta / \langle\mathcal{R}_{\tau\alpha}\rangle_\theta$ becomes even smaller than (26). But the nuclear burning occurs mainly in the central zone of the sun ($R \sim 0.1 R_\odot$). For example, the two–body reaction rate $\langle\mathcal{R}_{\tau\alpha}\rangle_\theta$ at the radius $0.1 R_\odot$ is already one order of magnitude less than at the centre [23]. Meanwhile, within the distance $R \lesssim 0.1 R_\odot$ from the solar centre the ratio $\langle\mathcal{R}_{e\tau\alpha}\rangle_\theta / \langle\mathcal{R}_{\tau\alpha}\rangle_\theta$ remains practically the same, $\sim 10^{-4}$. Therefore, in calculating the network of the solar *pp*–chain reactions one can ignore the non-radiative synthesis of ^{7}Be only if the required accuracy is less than $\sim 0.01\,\%$.

For more hot and dense stars such accuracy restrictions are more essential as the ratio

$$\frac{\langle \mathcal{R}_{e\tau\alpha}\rangle_\theta}{\langle \mathcal{R}_{\tau\alpha}\rangle_\theta} = \frac{\langle \Sigma_{e\tau\alpha}\rangle_\theta}{\langle \sigma_{\tau\alpha} v_{\tau\alpha}\rangle_\theta} n_e \tag{27}$$

is proportional to the electron density n_e. Furthermore, by comparing the data of Table I with the corresponding data for the two–body reaction given in Ref. [23], one can find that the ratio $\langle \Sigma_{e\tau\alpha}\rangle_\theta / \langle \sigma_{\tau\alpha} v_{\tau\alpha}\rangle_\theta$ increases with temperature.

Another theory in which the omission of the three–body non-radiative synthesis of ^{7}Be may be detrimental is that of the primordial nucleosynthesis during which practically all the nuclei were created when the temperature was in the range

$$0.3 \times 10^9\ {}^\circ\mathrm{K} < \theta < 3 \times 10^9\ {}^\circ\mathrm{K} \tag{28}$$

with the electron density being (see Ref. [3])

$$n_e \approx 1.5 \times 10^{29} \left(\frac{\theta}{10^9}\right)^{\frac{3}{2}} \exp\left(-\frac{5.93 \times 10^9}{\theta}\right)\ \mathrm{cm}^{-3}\ . \tag{29}$$

At $\theta = 10^9$ °K, for example, formula (29) gives

$$n_e \approx 0.40 \times 10^{27}\ \mathrm{cm}^{-3}\ , \qquad \theta = 10^9\,{}^\circ\mathrm{K}\ ,$$

which is one order of magnitude higher than the electron density (1) in the solar interior.

Using the data of Table I and the corresponding data for the two–body reaction (2) of Ref. [23], together with the electron density (29), we calculated the ratio (27) for the primordial nucleosynthesis. The temperature dependence of this ratio which we found for the temperature range (28) is given in Table III. It is seen that at the earliest stages of the nucleosynthesis, when the temperature was $\theta \approx 3\times 10^9$ °K, the non-radiative synthesis (3) of ^{7}Be nuclei was five times faster than for the two–body fusion (2). Thus, if the electron density given by formula (29) is not far from the actual density that existed at the early stages of the Universe, then three–body reactions of the type (3) must be taken into account in theories concerning primordial nucleosynthesis. It can make a differences since the uncertainty in the ^{7}Be production in the primordial nucleosynthesis is currently assumed to be of the order of 16% [24].

Despite the fact that in the solar plasma the contribution of the three–body process is rather small, in enegy balance considerations this might not be the case. In each event (3) or (2) an energy of 1.59 MeV is released. The energy produced by the radiative capture (2), is carried away by a photon which can penetrate the inner layers of the sun and even escape. In contrast, the electron which receives the energy produced in the reaction (3) remains in the interior of the sun and eventually distributes its excess energy among the other plasma particles. This causes an additional heating which is not taken into account by the standard model of the sun.

To evaluate the importance of three–body reactions in the energy balance quantitatively we calculated the rates of the energy released, due to the reactions (3) per cm^3 per second at different radial positions in the sun. These are presented in Table II. These results were obtained using the reaction rates of Table I together with the radial dependences of the particle densities and the temperature given in Tables 4.4 and 4.5 of Ref. [1]. By integrating

over the sun volume, we obtain the total energy generated in the sun every second from the non-radiative synthesis of ^{7}Be,

$$\left(\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t}\right)_{e\tau\alpha} = 0.582 \times 10^{16}\ \frac{\mathrm{erg}}{\mathrm{sec}}\ .$$

As compared to the total photon luminosity of the sun [1]

$$L_\odot = 0.386 \times 10^{34}\ \frac{\mathrm{erg}}{\mathrm{sec}}$$

it is rather small. Even during the whole period of its existence, $T_\odot \approx 4.55 \times 10^9$ years, the sun has generated, via the three–body reaction (3),

$$\left(\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t}\right)_{e\tau\alpha} T_\odot \approx 0.835 \times 10^{33}\ \mathrm{erg}\ ,$$

which is less than the energy radiated every second. This energy, however, has been stored inside the sun unless it was somehow transferred to the surface. In this connection the energy balance in the sun should, perhaps, be re-examined with inclusion not only the reaction (3) but also other nonradiative fusion reactions such as

$$\begin{aligned} e + p + d &\longrightarrow e + {}^3\mathrm{He}\ , \\ e + p + {}^7\mathrm{Be} &\longrightarrow e + {}^8\mathrm{B}\ , \\ e + p + {}^7\mathrm{Li} &\longrightarrow e + {}^8\mathrm{Be}\ , \\ &\text{etc.} \end{aligned}$$

In conclusion, we can say that the three–body reaction (3) contributes to ^{7}Be production in the sun only about 0.01%, it perhaps plays a role in the energy balance in the sun, and definitely it was important at the early stages of the primordial nucleosynthesis.

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TABLES

TABLE I. Temperature dependence of the non-radiative capture rate, $\langle \mathcal{R}_{e\tau\alpha} \rangle_\theta$, in units $\mathrm{cm}^6\mathrm{mole}^{-2}\mathrm{sec}^{-1}$.

θ (10^6 °K)	$\langle\Sigma\rangle N_A^2$	θ (10^6 °K)	$\langle\Sigma\rangle N_A^2$
1	4.653×10^{-55}	100	2.656×10^{-10}
2	3.920×10^{-43}	110	6.177×10^{-10}
3	2.363×10^{-37}	120	1.302×10^{-9}
4	1.039×10^{-33}	130	2.531×10^{-9}
5	4.034×10^{-31}	140	4.606×10^{-9}
6	3.797×10^{-29}	150	7.927×10^{-9}
7	1.427×10^{-27}	160	1.301×10^{-8}
8	2.839×10^{-26}	180	3.119×10^{-8}
9	3.549×10^{-25}	200	6.605×10^{-8}
10	3.122×10^{-24}	250	2.944×10^{-7}
11	2.087×10^{-23}	300	9.123×10^{-7}
12	1.121×10^{-22}	350	2.236×10^{-6}
13	5.034×10^{-22}	400	4.661×10^{-6}
14	1.950×10^{-21}	450	8.636×10^{-6}
15	6.671×10^{-21}	500	1.464×10^{-5}
16	2.053×10^{-20}	600	3.469×10^{-5}
18	1.499×10^{-19}	700	6.853×10^{-5}
20	8.289×10^{-19}	800	1.195×10^{-4}
25	2.538×10^{-17}	900	1.901×10^{-4}
30	3.432×10^{-16}	1000	2.826×10^{-4}
40	1.511×10^{-14}	1250	6.170×10^{-4}
50	2.208×10^{-13}	1500	1.104×10^{-3}
60	1.697×10^{-12}	1750	1.738×10^{-3}
70	8.606×10^{-12}	2000	2.507×10^{-3}
80	3.273×10^{-11}	2500	4.380×10^{-3}
90	1.009×10^{-10}	3000	6.583×10^{-3}

TABLE II. Non-radiative reaction rate, $\langle \mathcal{R}_{e\tau\alpha} \rangle_\theta$, its ratio to the corresponding radiative reaction rate, and the rate of the energy generation due to the non-radiative fusion, as functions of the radial position in the sun.

$R/R_\odot$	$\langle \mathcal{R}_{e\tau\alpha} \rangle_\theta$ cm^{-3}sec^{-1}	$\langle \mathcal{R}_{e\tau\alpha} \rangle_\theta / \langle \mathcal{R}_{\tau\alpha} \rangle_\theta$	ΔE erg/(cm^{3}sec)
0.00	7148	0.294×10^{-3}	0.182×10^{-13}
0.01	7110	0.288×10^{-3}	0.181×10^{-13}
0.02	6387	0.282×10^{-3}	0.162×10^{-13}
0.03	5416	0.272×10^{-3}	0.138×10^{-13}
0.04	4139	0.260×10^{-3}	0.105×10^{-13}
0.05	3202	0.246×10^{-3}	0.814×10^{-14}
0.06	2194	0.230×10^{-3}	0.558×10^{-14}
0.07	1407	0.213×10^{-3}	0.358×10^{-14}
0.08	810	0.196×10^{-3}	0.206×10^{-14}
0.09	515	0.180×10^{-3}	0.131×10^{-14}
0.10	270	0.163×10^{-3}	0.688×10^{-15}
0.15	10	0.962×10^{-4}	0.258×10^{-16}
0.20	0.241	0.517×10^{-4}	0.612×10^{-18}
0.30	0.334×10^{-4}	0.123×10^{-4}	0.850×10^{-22}
0.40	0.126×10^{-9}	0.274×10^{-5}	0.321×10^{-27}
0.50	0.194×10^{-14}	0.643×10^{-6}	0.493×10^{-32}

TABLE III. Temperature dependence of the ratio of the three– to two–body reaction rates for the primordial nucleosynthesis.

θ (10^6 °K)	$\langle \mathcal{R}_{e\tau\alpha} \rangle_\theta / \langle \mathcal{R}_{\tau\alpha} \rangle_\theta$
300	0.181×10^{-8}
350	0.404×10^{-7}
400	0.428×10^{-6}
450	0.275×10^{-5}
500	0.124×10^{-4}
600	0.124×10^{-3}
700	0.671×10^{-3}
800	0.244×10^{-2}
900	0.684×10^{-2}
1000	0.159×10^{-1}
1250	0.766×10^{-1}
1500	0.231
1750	0.530
2000	1.02
2500	2.70
3000	5.44

FIGURES

FIG. 1. Jacobi vectors in the configuration and momentum space.

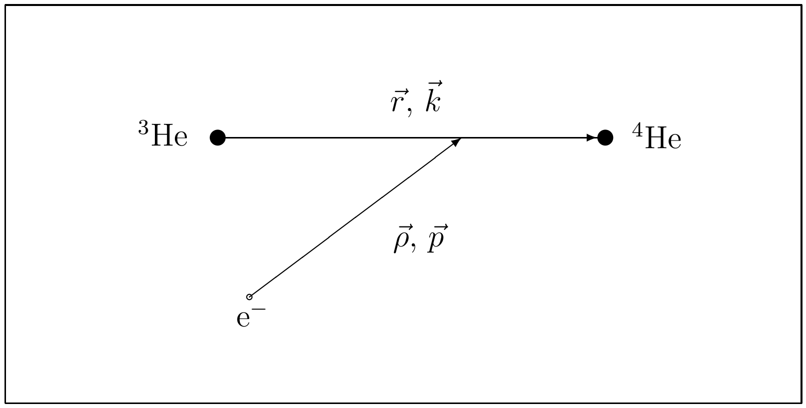

FIG. 2. Low–lying energy levels (in MeV) of ^{7}Be nucleus and the ^{3}He-^{4}He threshold energy.

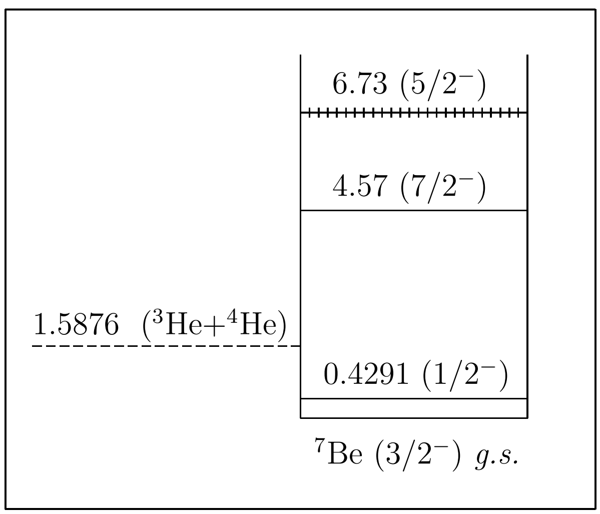

FIG. 3. Effective ^{3}He-^{4}He potential for three different states corresponding to relative orbital angular momentum $\ell = 0, 1$ and total angular momentum $J = 1/2$, $3/2$.